Efficient evolutionary dynamics with extensive-form games

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Game theory

- Games describe strategic-interaction situations in which each agent behaves rationally maximizing her utility.
- Game theory provides:
  - models (e.g., strategic-form vs. extensive-form)
  - solution concepts (e.g., Nash equilibrium)
- Game theory assumes agents to be rational and to have common knowledge about the game (i.e., payoffs).
Evolutionary game theory

- Evolutionary game theory drops the assumption of rationality of agents
- Agents continuously adapt their strategies
- Strategies evolve in time according to a dynamical equation (e.g., replicator dynamics)
- Steady states are Nash equilibria (except for the boundary)
- Most works on evolutionary game theory focus on strategic-form games
EGT and multi-agent learning

• Evolutionary game theory is strictly connected to multi-agent learning
• Q-learning and SARSA dynamics (in expectation) can be formally described by using replicator dynamics with mutation [Tuyls, Hoen, Vanschoenwinkel, 2006]
• Evolutionary game theory tools are sometimes used as heuristics in Nash-finding algorithms
Extensive-form games

![Game Tree Diagram]

\[ u \in \{L_1, R_1\} \]
\[ \Pi = \pi_1 \times \pi_2 \]
\[ N,A,V,T, \]

\[ \iota^i \in \sum, \text{with abuse of notation, we write} \]

\[ \text{Extensive-form game definition} \]

\[ \text{Plan of agent} \]

\[ \text{Game theoretical preliminaries} \]

\[ \text{normal form by} \]

\[ \text{Obtained from the normal form by} \]

\[ \text{Example 1} \]

\[ \text{The reduced normal form of the game in Fig. 1} \]

\[ \text{Non-terminal sequences} \]

\[ \text{Sequence-form strategy of agent} \]

\[ \text{with} \]

\[ \text{Every sequence} \]

\[ \text{Is a} \]
Extensive-form games: normal form

\[
\begin{array}{c|cc}
\text{agent 1} & \text{agent 2} & \text{r} \\
\hline
L_1^{*} & 2,4 & 2,4 \\
R_1L_2L_3 & 3,1 & 2,1 \\
R_1L_2R_3 & 3,1 & 4,2 \\
R_1R_2L_3 & 3,3 & 2,1 \\
R_1R_2R_3 & 3,3 & 4,2 \\
\end{array}
\]

\[
\pi_1 = \begin{cases} 
\pi_{1,L_1^{*}} = \frac{1}{3} \\
\pi_{1,R_1L_2L_3} = 0 \\
\pi_{1,R_1L_2R_3} = \frac{1}{3} \\
\pi_{1,R_1R_2L_3} = 0 \\
\pi_{1,R_1R_2R_3} = \frac{1}{3} 
\end{cases}
\]

\[
\pi_2 = \begin{cases} 
\pi_{2,L} = 1 \\
\pi_{2,r} = 0 
\end{cases}
\]
Extensive-form games: normal form

\[\begin{array}{c|cc}
\text{agent 1} & \text{l} & \text{r} \\
\hline
\text{L}_1^* & 2,4 & 2,4 \\
\text{R}_1\text{L}_2\text{L}_3 & 3,1 & 2,1 \\
\text{R}_1\text{L}_2\text{R}_3 & 3,1 & 4,2 \\
\text{R}_1\text{R}_2\text{L}_3 & 3,3 & 2,1 \\
\text{R}_1\text{R}_2\text{R}_3 & 3,3 & 4,2 \\
\end{array}\]

\[\pi_1 = \begin{cases} 
\pi_{1,L_1^*} = \frac{1}{3} \\
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\pi_{1,R_1L_2R_3} = \frac{1}{3} \\
\pi_{1,R_1R_2L_3} = 0 \\
\pi_{1,R_1R_2R_3} = \frac{1}{3} 
\end{cases} \]

\[\pi_2 = \begin{cases} 
\pi_{2,l} = 1 \\
\pi_{2,r} = 0 
\end{cases} \]
Extensive-form games: normal form

\[ \pi_1 = \begin{cases} \pi_{1,*} = \frac{1}{3} \\ \pi_{1,1} = 0 \\ \pi_{1,2} = \frac{1}{3} \\ \pi_{1,3} = 0 \end{cases} \]

\[ \pi_2 = \begin{cases} \pi_{2,1} = 1 \\ \pi_{2,2} = 0 \end{cases} \]
Extensive-form games: normal form

Normal form is exponential in the size (number of information sets, actions, outcomes) of the game tree

\[
\pi_1 = \begin{cases} 
\pi_{1,L_1^*} = \frac{1}{3} \\
\pi_{1,R_1L_2L_3} = 0 \\
\pi_{1,R_1L_2R_3} = \frac{1}{3} \\
\pi_{1,R_1R_2L_3} = 0 \\
\pi_{1,R_1R_2R_3} = \frac{1}{3} 
\end{cases}
\]

\[
\pi_2 = \begin{cases} 
\pi_{2,l} = 1 \\
\pi_{2,r} = 0 
\end{cases}
\]
Evolutionary dynamics and normal form

Replicator dynamics is applied directly to the normal form

<table>
<thead>
<tr>
<th>agent 1</th>
<th>l</th>
<th>r</th>
</tr>
</thead>
<tbody>
<tr>
<td>L₁*</td>
<td>2,4</td>
<td>2,4</td>
</tr>
<tr>
<td>R₁L₂L₃</td>
<td>3,1</td>
<td>2,1</td>
</tr>
<tr>
<td>R₁L₂R₃</td>
<td>3,1</td>
<td>4,2</td>
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<td>4,2</td>
</tr>
</tbody>
</table>

\[
\dot{\pi}_1(p) = \pi_1(p) \cdot [(e_p - \pi_1)^T \cdot U_1 \cdot \pi_2]
\]

\[
\dot{\pi}_2(p) = \pi_2(p) \cdot [\pi_1^T \cdot U_2 \cdot (e_p - \pi_2)]
\]

\(e_p\) is a vector of zeros except 1 in position \(p\)

If the starting point is **fully mixed** or a **small mutation** term is present, the replicator dynamics converges to a **subgame perfect equilibrium** with **perfect information games**

**Drawback:** the number of dynamical equations is equal to the number of actions \(p\), that is exponential in the size of the game tree
Extensive-form games: sequence form

\[
x_1 = \begin{bmatrix}
1 \\
\frac{1}{3} \\
\frac{1}{3} \\
\frac{1}{3}
\end{bmatrix}
\]

\[
x_2 = \begin{bmatrix}
1 \\
1 \\
0 \\
0
\end{bmatrix}
\]

\[
x_i(q_{\emptyset}) = 1
\]

\[
x_i(q) = \sum_{a \in \rho(w)} x_i(q|a)
\]
Extensive-form games: sequence form

\[
x_i(q_{\emptyset}) = 1 \quad \text{and} \quad x_i(q) = \sum_{a \in \rho(w)} x_i(q|a)
\]
Extensive-form games: sequence form

\[
x_i(q_{\emptyset}) = 1
\]
\[
x_i(q) = \sum_{a \in p(w)} x_i(q|a)
\]
Extensive-form games: sequence form

\[ x_i(q_{\emptyset}) = 1 \]
\[ x_i(q) = \sum_{a \in \rho(w)} x_i(q|a) \]

\[
\begin{array}{c|cc}
\text{agent 1} & q_{\emptyset} & l & r \\
\hline
q_{\emptyset} & & & \\
L_1 & 2, 4 & & \\
R_2 & & & \\
(R_1, L_2) & 3, 1 & & \\
R_1, R_2 & 3, 3 & & \\
R_1, L_3 & 2, 1 & & \\
R_1, R_3 & 4, 2 & & \\
\end{array}
\]

\[
x_1 = \begin{bmatrix}
1 \\
\frac{1}{3} \\
\frac{1}{3} \\
\frac{1}{3}
\end{bmatrix}
\]
\[
x_2 = \begin{bmatrix}
1 \\
1 \\
0 \\
0
\end{bmatrix}
\]
Extensive-form games: sequence form

Game theoretical preliminaries:

Extensive-form game definition:

Although reduced normal form can be much smaller than the sequences. A sequence can be constituted by a tabular and a set of constraints. Sequence form actions are called extensive–form game, perfect–information game and apartition .

\[ V \rightarrow w \]

is the set of plans of agent \( w \), \( \rho \), \( L \), \( H \) returns the agent that acts at a given decision node.

Example 2

An imperfect–information game is some action \( q \) that is true if sequence \( R \) is possible. The sequence form is linear in the size of the game tree.

\[ \chi = \begin{pmatrix} 1 \\ \frac{1}{3} \\ \frac{4}{3} \end{pmatrix} \quad x_1 = \begin{bmatrix} 1 \\ \frac{1}{3} \\ \frac{4}{3} \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \]

\[ x_i(q) = \sum_{a \in \rho(w)} x_i(q|a) \]

\[ x_i(q_\emptyset) = 1 \]

\[ x_i(q) = \sum_{a \in \rho(w)} x_i(q|a) \]
Extensive-form games: sequence form

Example 1

\[ x_i(q_\emptyset) = 1 \]

\[ x_i(q) = \sum_{a \in p(w)} x_i(q|a) \]
Extensive-form games: sequence form

Sequence form is linear in the size of the game tree, but it presents constraints.
Evolutionary dynamics and sequence form

If the standard replicator dynamics is applied to the sequence form (considering $q$ as $p$), some issues are in place

$$
\begin{array}{c|cc|c}
\text{agent 1} & q_\emptyset & l & r \\
\hline
q_\emptyset & 2,4 &  & \\
L_1 & 3,1 &  & \\
R_1 & 3,3 &  & \\
R_1L_2 & 2,1 &  & \\
R_1R_2 & 4,2 &  & \\
\end{array}
$$

$$
x_1 = \begin{bmatrix}
1 \\
\frac{1}{3} \\
\frac{4}{3} \\
\frac{4}{3} \\
\frac{4}{3} \\
\frac{4}{3}
\end{bmatrix}
\quad x_2 = \begin{bmatrix}
1 \\
1 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\quad x_i(\emptyset) = 1
\quad x_i(q) = \sum_{a \in \rho(w)} x_i(q|a)
$$

The application of the standard replicator dynamics is not consistent due to the sequence-form constraints

Starting from

$$
x_1 = \begin{bmatrix}
1 \\
\frac{1}{3} \\
\frac{4}{3} \\
\frac{4}{3} \\
\frac{4}{3} \\
\frac{4}{3}
\end{bmatrix}
\quad x_2 = \begin{bmatrix}
1 \\
1 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
$$

We obtain

$$
x_1^{T}(t+1) = \begin{bmatrix}
0 \\
\frac{1}{3} \\
\frac{1}{6} \\
\frac{1}{6} \\
0 \\
0
\end{bmatrix}
\quad x_2^{T}(t+1) = \begin{bmatrix}
\frac{1}{2} \\
\frac{1}{2} \\
0
\end{bmatrix}
$$

Then

$$
x_i(\emptyset, t+1) \neq 1
$$
Replicator dynamics for sequence form

\[ x_1(q, t + 1) = x_1(q, t) \cdot \frac{g_q^T(x_1(t)) \cdot U_1 \cdot x_2(t)}{x_1^T(t) \cdot U_1 \cdot x_2(t)} \]

\[ x_2(q, t + 1) = x_2(q, t) \cdot \frac{x_1^T(t) \cdot U_2 \cdot g_q(x_2(t))}{x_1^T(t) \cdot U_2 \cdot x_2(t)} \]
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\]
Properties

• **Theorem** Given a well-defined sequence-form strategy at time $t$, our replicator dynamics generates a well-defined sequence-form strategy at time $t+1$.

• **Corollary** For every starting point, our replicator dynamics moves in the space of sequence-form strategies.

• **Theorem** Given:
  – a normal-form strategy and
  – its realization-equivalent sequence-form strategy at time $t$,
our replicator dynamics generates a sequence-form strategy at time $t+1$ that is realization equivalent to the normal-form strategy generated by the standard replicator dynamics.

• **Corollary** For every starting point, our replicator dynamics is realization equivalent to the standard replicator dynamics.

• **Remark** Our replicator dynamics exponentially reduces the spatial (and time) complexity of the standard replicator dynamics.
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Additional results

• We extended the previous results to the continuous case
  – well-defined sequence-form strategies
  – realization equivalence

• We discuss how our replicator dynamics can be adopted to study the stability of strategies by means of Jacobian
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Conclusions and future works

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  – Proof of realization equivalence
  – Exponential reduction of the size of the replicator dynamics

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